An Analog Computer Primer

Brian L. Stuart

August 21, 2007

After examining mechanical analog computing devices and looking at a demonstration of a special purpose electronic analog computer, it would be helpful to examine some aspects of general purpose electronic analog computers. These notes are meant to serve that purpose. We will take a quick look at how the computing components operate and then look at an example problem to be solved on such a computer.

1 The First Two Hours of an Electrical Engineering Cirriculum

Before we can understand how the computing components work, we need to review a few facts about the flow of electrical current. Electrical current is the result of electrons (e^-) flowing from a point with a more negative potential to a point with a more positive potential. Current (denoted by the letter i) flows, by definition, from the higher voltage point to the lower point in a circuit. (Note that the conventional direction of current is opposite that of the electron flow.) In analyzing a circuit, we generally ascribe an arbitrary direction to the current and denote it with an arrow. If the actual current flows in the same direction as we've labeled, then it's magnitude is positive, otherwise, it is negative. Current is given in units of Amperes, but we'll nearly always be dealing in much smaller units of milliamps (mA).

In many circuits, there are points where several branches of the circuit come together. Because current is based on electron flow and because of the conservation of mass principle, the current flowing into such a point must equal that flowing out of it. In other words, the sum of all currents flowing into any point in a circuit must equal zero.

The next thing we need to understand is a device called a *resistor.* As its name implies, a resistor resists the flow of current. In particular, the current through a resistor is proportional to the voltage across it and inversely proportional to its resistance value. The value of a resistor is given in units called Ohms (denoted Ω). We will often be dealing with relatively large resistance values measured in kilohms (KΩ) or megohms (MΩ). Given the current direction

relative to the voltage polarity shown in Figure 1, the current is given by
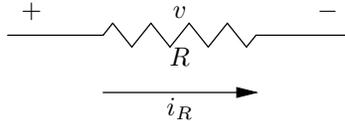
$$i_R = \frac{v}{R}.$$

Figure 1: Voltage and Current in a Resistor

The next important component we will be using is the *capacitor*. Capacitance refers to the ability to store electrical charge in such a device. The storage value of a capacitor is measured in Farads, but real capacitors are nearly always a tiny fraction of a full Farad. We will usually be dealing in micro-farads (μF). Current through a capacitor is proportional to the capacitance and to the first derivitive of the voltage across it. In Figure 2, the current is given by
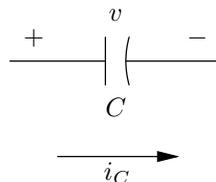
$$i_C = C\frac{dv}{dt}.$$

Figure 2: Voltage and Current in a Capacitor

2   Operational Amplifiers

The term *operational amplifier* was coined in 1947 to describe differential amplifiers of high gain capable of amplifying DC signals as well as AC signals. The differential nature of these amplifiers simply means that we're amplifying the difference between two input voltages. We will show a generic operational amplifier (op-amp) as in Figure 3. In our analysis of this amplifier, we will make the following simplifying approximations to the model:

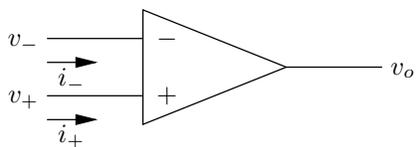1. The gain is infinite. That is "$\frac{v_o}{(v_+ - v_-)} = \infty$."

Figure 3: An Operational Amplifier

2. Since no real device can produce infinite output voltage, this means that the differential input voltage must be zero. Or $v_+ = v_-$. (Hence, the quotes on the equation describing the gain.)

3. Furthermore, we will assume that the current in both the positive and negative input circuits will be zero ($i_+ = i_- = 0$).

3 Computational Circuits Using Op-Amps

The model of an op-amp we just examined allows us to do a number of things when we connect resistors and capacitors to it. We'll be looking at multiplying a signal by a constant value, performing a weighted sum and integration.

3.1 Voltage Divider

First, however, we'll look at a simple circuit composed of only resistors as shown in Figure 4.
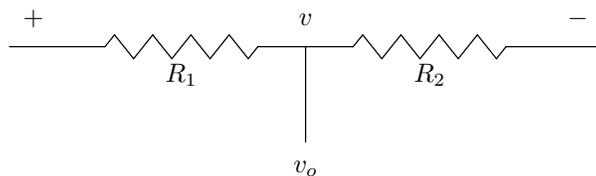

Figure 4: Voltage Divider

In this circuit, we'll assume that no current flows out of the tap between the two resistors. (This assumption, as many in engineering are, is not always justifiable. We simply keep that in mind and correct for any inaccuracies if they are significant.) Given that assumption, we can see that the current through both resistors is equal and is $i = v/(R_1 + R_2)$. The output voltage is then given by $v = iR$ which in this case gives

$$v_o = v\frac{R_2}{R_1 + R_2}$$

Notice that the constant by which we are performing multiplication is strictly less than one. This restriction is not present in the next computational circuit.

We can create an adjustable voltage divider by using a potentiometer which is usually implemented by a strip of carbon with a wiper. The ends of the resistive strip are connected to the voltage we want to divide. Then the wiper becomes the point between our two resistors and gives a voltage proportional to the voltage to be divided and to the fraction of the strip which is "below" the wiper.

3.2 Multiplication by a Constant
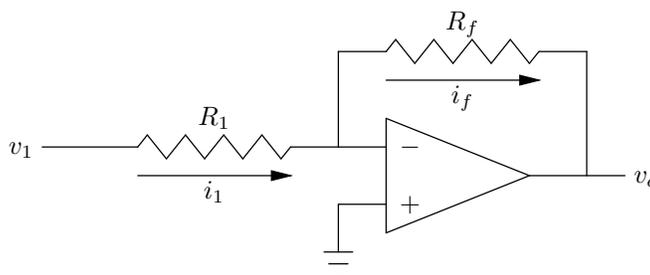
Consider the circuit shown in Figure 5.

Figure 5: Scaling the Magnitude of a Signal

Because no current flows into the op-amp and because all current at a node must sum to zero, the current entering the input leg must be equal to that flowing throught the feedback leg from the input to the output ($i_1 = i_f$). We also know that the voltage at the inverting input must be zero (that is the voltage indicated by the ground shown connected to the non-inverting input). From our properties of resistors, then, we know that

$$i_1 = \frac{v_1}{R_1} = i_f = -\frac{v_o}{R_f}.$$

Consequently,

$$v_o = -v_1\frac{R_f}{R_1}$$

which means that we're multiplying the magnitude of the input voltage by the constant R_f/R_1.

Notice that the sign of the voltage has changed. This is a universal trait of the type of op-amp circuits we'll be using in analog computers. As we'll see in the example, it doesn't cause any significant problems in using the computer.

3.3   Weighted Addition

Weighted addition of signals entails only a slight extension to our previous circuit and analysis. The new circuit is shown in Figure 6.
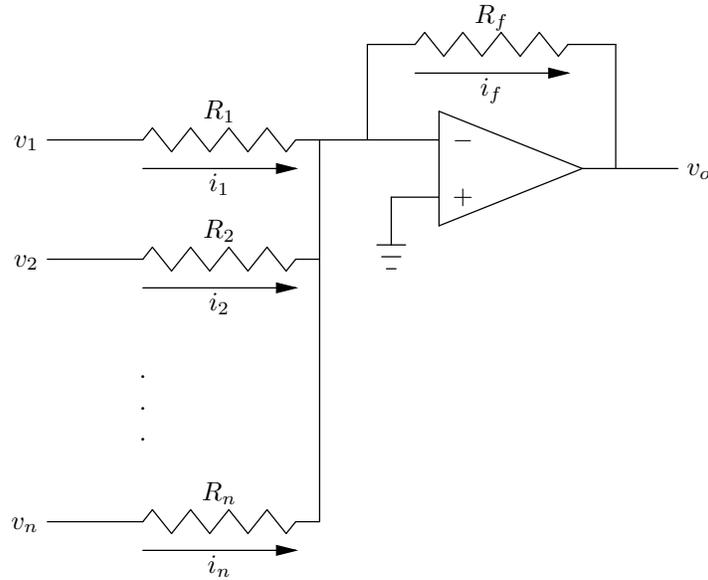

Figure 6: A Weighted Adder

Here notice that since the same ground rules for op-amps apply, the feedback current must equal the sum of the input currents. (Remember that some of these currents may be negative.)

$$i_f = \sum_{k=1}^{n} i_k = \sum_{k=1}^{n} \frac{v_k}{R_k} = -\frac{v_o}{R_f}$$

and

$$v_o = -\sum_{k=1}^{n} v_k \frac{R_f}{R_k}.$$

3.4   Integration

In this subsection, we'll only look at integrating a single input signal. However, the circuit and analysis given here can be extended as above to create a circuit that will integrate a weighted sum. The integrator is shown in Figure 7.
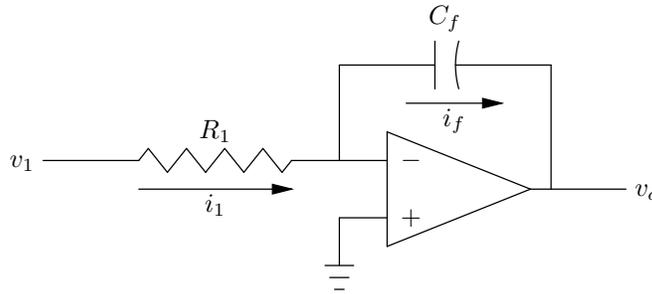
Figure 7: An Integrator

Following the same type of analysis given above, the equating of input and feedback currents gives

$$i_1 = \frac{v_1}{R_1} = i_f = -C_f \frac{dv_o}{dt}.$$

Multiplying by $-1/C_f$ and integrating both sides gives

$$v_o = -\frac{1}{R_1 C_f} \int v_1 dt.$$

Those in whose minds Calculus is fresh will notice that we've been a bit sloppy here. It is much more accurate to write

$$v_o(t) = -\frac{1}{R_1 C_f} \int_{t_0}^{t} v_1(t) dt + c$$

where c is an *initial condition* and t_0 is the point in time when the integrator goes into operation. We will usually take $t_0 = 0$ and always discuss time relative to that "beginning of time." We'll have to be more careful if the computer causes itself to be reset with new initial conditions at various points during the computation. In those cases, we'll call the beginning of the whole computation time 0 and talk in relative terms about the intervals between resets.

4   Electronic Analog Computers

While these components are not all that are found on real electronic analog computers (those few that still remain), they are the primary ones. We will denote them with the commonly used symbols shown in Figure 8.
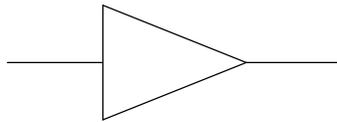
Notice that we haven't specified what gain factors are on the inputs of the summers or integrators. In practice, we usually have a small variety of gains available; usually 1, 4 and 10. The gain we use is noted inside the symbol beside
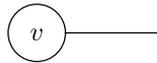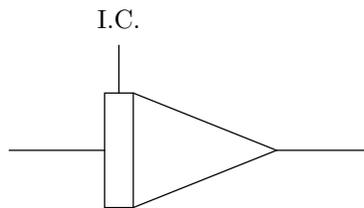
Constant Voltage Source
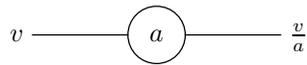
Voltage Divider

Weighted Adder

Integrator

Figure 8: Symbols used for Computing Components

the input in question. For the example in the next section, we use only gain 1 inputs, so we'll just leave them off completely.

Also notice the input for initial conditions shown at the top of the integrator. This is a common convenience feature. From the equation give for integration, it is clear that we can actually add the initial condition in a summer in a stage following the integrator. However, most real computers provide initial condition inputs on the integrators.

5   An Example

A frequently discussed example in differential equations and in analog computers is a mass suspended from a spring where there is viscous damping. There are three forces acting on such a mass. The gravitational force is given by

$$F_g = mg$$

where g is the gravitational constant. The spring exerts a force to return the mass to a neutral state which is proportional to the distance away from the neutral position

$$F_s = -Ky$$

where $y = 0$ is the neutral postion for the spring. While the mass is in motion, there is a force acting against that motion and proportional to the velocity:

$$F_d = -d\frac{dy}{dt}.$$

From Newton's laws of motion, we know that

$$F = ma$$

which, in this case, gives us

$$F_g + F_s + F_d = m\frac{d^2y}{dt^2}.$$

Solving for the highest order derivative, we have

$$\frac{d^2y}{dt^2} = -\frac{d}{m}\frac{dy}{dt} - \frac{K}{m}y + g.$$

Once we express our differential equations in this form, we now create a circuit of computing elements that solves the equations. We assume the existance of a voltage (or a set of voltages that sum to one) proportional to the highest order derivative. Once we have that, we can easily create a voltage proportional to the negative of the next lower derivative by passing the higher order one through an integrator. We chain a number of integrators together to create the
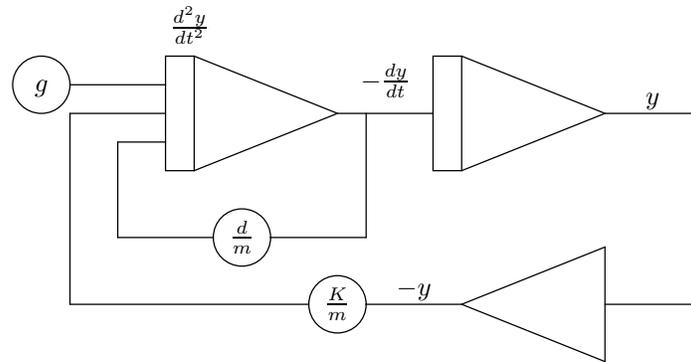
8

Figure 9: Solution to the Spring and Mass Problem

variable itself. For this second-order equation, we need two integrators. Now that we have each of the derivatives, we construct the weighted sum that is our highest order deravitive and feed that into the input of the first integrator.

The spring and mass equation gives a circuit shown in Figure 9.

Frequently, the constants of a problem would yeild difficulties if we just applied this procedure for solution. For example, the various values might experience fluctuations that are so slow that we waste time simulating them in real time. Or, more serious, the fluctuations might be so fast that the op-amp circuits have problems operating correctly or that our output devices can't keep up. Similarly, the range of values for the various points in a circuit might go out of range for our devices or they may alway be small increasing the susceptability to error. In practice, we use several rules-of-thumb to estimate behaviors such as these and create scaling factors that make the range of values usable. The effects of such scaling are most easily seen by way of a substitution of the troublesome variable by a constant times a new variable. These substitutions are done prior to the development of the circuit.